POWER LAWS AND SCALING LAWS IN EARTHQUAKE OCCURRENCE

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1. Size Distributions, Power Laws & SOC:
2. Waiting-Time Distributions & Scaling Laws:
Traditional Reductionist Way of Doing

Case of physics:

- Matter is complex

⇒ Find its ultimate constituents

Case of earthquakes:

- An earthquake is a very complex phenomenon whose physics is largely unknown

⇒ Study specific parts of the problem

Great 2004 Sumatra-Andaman earthquake: more than 100 papers! (by title)

15 in *Nature* or *Science*!
0. Introduction

Complementary Approach: Complex-Systems Philosophy

- Can we learn something from collective properties?

⇒ Study emergent statistical properties of (relatively) large areas:
   Concentrate on the whole rather than on the parts

Hundreds of earthquakes are needed for a single paper!
1. Size Distributions, Power Laws & SOC:

**Gutenberg-Richter Law**: most important law for the statistics of seismicity

- For each earthquake with magnitude $M \geq 8$ there are about
  - 10 with $M \geq 7$
  - 100 with $M \geq 6$, etc...

⇒ Number of earthquakes decays exponentially

$$N(M) \propto 10^{-bM}$$

(with $b \approx 1$)

⇒ Many small earthquakes, few big, good news!

Distribution of magnitudes

- We use the concept of probability density, defined as

\[ D(M) \equiv \frac{\text{Prob}[M \leq \text{magnitude} < M + dM]}{dM} \]

and estimated as

\[ D(M) = \frac{\text{number of earthquakes with } M \leq \text{magnitude} < M + dM}{\text{total number of earthquakes} \times dM} \]

\[ \Rightarrow D(M) \propto dN(M)/dM \]

- The Gutenberg-Richter law yields the same function for \( D(M) \)

\[ D(M) \propto 10^{-bM} \propto e^{-b\ln 10 M} \]
• Earthquake radiated energy: energy is (roughly) an exponential function of magnitude, \( E \propto 10^{1.5M} \)

As \( D(E)\,dE = D(M)\,dM \Rightarrow D(E) = D(M)\,dM/dE \)

\( \Rightarrow \) Energy follows a power-law distribution: \( D(E) \propto 1/E^{1+0.67b} \)
Power Laws and Scale Invariance

What is special about power laws?

• Let us perform a scale transformation on a function $y = F(x)$,

\[ x \rightarrow x' \equiv ax, \]
\[ y \rightarrow y' \equiv cy. \]

In the new axes, the function $F(x)$ transforms into

\[ F(x) \rightarrow cF(x'/a) \]

• Scale invariance means that the new function looks the same, $F(x) = cF(x/a)$.

The solution is given by a power law:

\[ F(x) = Ax^\alpha \quad \text{with} \quad c = a^\alpha \quad A \text{ arbitrary,} \]
Invariance of power laws under scale transformations
• Invariance of power laws under scale transformations
Example of scale invariance: **fractals**

Fractal: an object that shows *the same structure at all scales*

No characteristic scale  ⇒  **Power-law distribution** of structure sizes
Scale Invariance of Earthquake Sizes

- Power law or “fractal” distribution of earthquake sizes (energy)
  ⇒ There is no characteristic size for earthquakes
  ⇒ It is not possible to answer this simple question:
    
    “How big are earthquakes in a given region?”
Mean Energy of Earthquakes

- Using the Gutenberg-Richter law, the mean energy:

\[ \langle E \rangle = \int_{\text{min}}^{\infty} ED(E)dE \propto \int_{\text{min}}^{\infty} \frac{dE}{E^{0.66}} = \infty \]

The mean radiated energy is infinite!

How can it be? The Earth has a finite energy content...
What does it mean?

\[ E(\text{Joules}) \approx 60000 \cdot 10^{1.5M} \text{ (roughly)} \]

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Energy (Joules)</th>
<th>Number</th>
<th>Total energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( 2 \times 10^{12} )</td>
<td>1000</td>
<td>( 2 \times 10^{15} )</td>
</tr>
<tr>
<td>6</td>
<td>( 6 \times 10^{13} )</td>
<td>100</td>
<td>( 6 \times 10^{15} )</td>
</tr>
<tr>
<td>7</td>
<td>( 2 \times 10^{15} )</td>
<td>10</td>
<td>( 2 \times 10^{16} )</td>
</tr>
<tr>
<td>8</td>
<td>( 6 \times 10^{16} )</td>
<td>1</td>
<td>( 6 \times 10^{16} )</td>
</tr>
</tbody>
</table>

In practice, the mean does not converge

⇒ This means that extreme (rare) events determine the dissipation

⇒ Big earthquakes are responsible of the energy release!

⇒ Bad news!!
The GR law is telling us something about the physics of earthquakes
But what?

- “Domino theory”

Tectonic fault: analogous to a domino-like network
Earthquake: chain reaction of topplings or avalanche
• **Branching process**

Each mother leaves a random number \( n \) of daughters

Which will be the total number of offsprings?

i.e., domino topplings \( \equiv \) "activity", proportional to energy
- Probability distribution of total activity, $\langle n \rangle < 1$

At the end, the activity dies
- Probability distribution of total activity, also for $\langle n \rangle > 1$

Finite probability of infinite activity
• Probability distribution of total activity, including $\langle n \rangle = 1$

All sizes are possible, power-law distribution
⇒ Power law distributions are very difficult to achieve ($\langle n \rangle = 1$)

- After any toppling, we don’t know what will happen next
- The perturbation propagates or not to give rise a catastrophic event depending on a huge number of microscopic details which are intrinsically out of control
- Consequences for predictability?
Another Example: Critical Points of Thermodynamic Phase Transitions

Magnetic material: atom = spin with 2 states

There exists a critical temperature $T_c$

- Above $T_c$: no magnetization, small clusters
- Below $T_c$: magnetization, one very large cluster
- At the precise value $T = T_c \Rightarrow$ clusters of all sizes $\Rightarrow$ power law!
Self-organized criticality

- How is the required fine tuning achieved?

- The power-law response emerges as a consequence of the attraction of the dynamics towards a critical point $\Rightarrow$ sandpile paradigm

- **Sandpile metaphor**
  - If there are few grains (flat pile) $\Rightarrow$ small avalanches, pile grows
  - If there are many grains (steep pile) $\Rightarrow$ large avalanches, pile decreases

  This mechanism makes the slope of the pile fluctuate around the critical state.

Bak et al. PRL 1987
Rockfalls

- Size measured in volume of rocks
  - Purple color: earthquake-triggered rockslide event in Umbria (Italy) in 1997
  - Green color: rockfalling at Yosemite (USA) from 1980 to 2002

Exponent 1.1

- Other similar phenomena:
  - Landslides
  - Snow avalanches
  - Sediment gravity flows in the oceans

Malamud, Phys. World 2004
Rice-pile avalanches

Frette et al. Nature 1996
Forest fires: Fires at Ontario (Canada), 1976–1996 (15308 fires)

- Size measured as burned area
- Trees store energy which is rapidly released by fire
- Conclusion: Forests have the largest number of trees allowed by fires

Volcanic eruptions

- Area covered by lava flows in the Springerville volcanic field, Arizona (USA) between 2.1 Myear and 0.3 Myear ago.

Cumulative number of eruptions versus area in km$^2$

Lahaie and Grasso, JGR 1998
Biological extinctions (?)

- Extinction measured as the percentage of extinct families in periods of 4 million years

Sepkoski, Paleobio. 1993; Raup, Bad Genes... 1991, shown in Bak 1996
Tropical cyclones (hurricanes):

- Dissipated Energy (PDI) of North Western Pacific typhoons, 1986–2007

Power-law distribution: \( D(PDI) \propto \frac{1}{PDI} \)

A. Ossó et al. preprint 2009
Rainfall: measured at one point of the Baltic coast, Jan-Jul 1999

- A rain event is defined as the continuous occurrence of rain between drought periods of minimum 1 minute.

- Dynamics:
  - Solar radiation provides energy
  - Evaporated water stores it
  - If a saturation threshold is reached ⇒ rain

Peters et al. PRL 2002
Summarizing:

- SOC $\Rightarrow$ sandpile dynamics and power-law distributions
- Many natural disasters $\Rightarrow$ sandpile-like dynamics and power-law distributions

Does this mean that the previous natural disasters are SOC?

Or something else is necessary (?)
1. Size Distributions, Power Laws & SOC:

First Confirmation of Self-Organized Criticality?

- **Rain:** there exist a critical point and the system is attracted close to it!

![Graph showing precipitation occurrence probability and variance](image-url)

2. Waiting-Time Distributions & Scaling Laws: Motivation

- For earthquakes (and others): are there other indications of criticality?
- Do power-law size distributions reflect some degree of self-similarity in time?
  
  1 year of earthquakes with $M \geq 5 \Leftrightarrow$
  10 years of earthquakes with $M \geq 6$ etc.?

- Complex-System philosophy:

  - Difficulties studying faults
    - Interaction between faults, no isolated faults exists
    - Problems assigning earthquakes to faults
    - Ambiguity to identify and even define faults
      ⇒ Study spatially extended areas

  - All earthquakes constitute a unique process
    ⇒ Do not distinguish between mainshocks, aftershocks, etc.

  ⇒ The robustness of the results will corroborate the coherence of the approach

Bak et al. PRL 2002
Waiting times

- Consider a fixed spatial region
- Consider earthquakes with magnitude larger than a threshold, \( M \geq M_c \)

Compute waiting time as the time between consecutive earthquakes

\[
\tau_i \equiv t_i - t_{i-1}
\]

\[ i = 1, 2, 3 \ldots \]

\( \Rightarrow \text{Broad scale of times} \Rightarrow \text{Gutenberg-Richter gives a poor description!} \)
Worldwide seismicity for $M \geq 5$, from 1973 to 2002
Worldwide seismicity for $M \geq M_c$, with $M_c$ variable, from 1973 to 2002.
Scale transformation of the axes

\[ \tau \longrightarrow R_c \tau \]

\[ D(\tau, M_c) \longrightarrow D(\tau, M_c)/R_c \]

with \( R_c(M_c) \) the rate of seismic activity: number of earthquakes per unit time

Scaling law:

\[ D(\tau, M_c) = R_c f(R_c \tau) \]
2. Universal scaling law for inter-event time distribution

Scaling function:

\[ f(\theta) \propto \frac{1}{\theta^{0.3}} e^{-\theta/1.4} \]

\[ \theta \equiv R\tau \]
Two main properties:

- Clustering

\[ f(\theta) \propto \frac{1}{\theta^{0.3}} e^{-\theta/1.4} \]

It is valid independently of the fit of \( f(\theta) \)

- Scaling

\[ D(\tau, M_c) = R_c f(R_c \tau) \]

In fact, the existence of clustering is clear before rescaling
Poisson process

- A dice decides if an earthquake happens or not
  - The dice has many faces
    (probability of occurrence very small, \( p \to 0 \))
  - The dice is thrown continuously in time, \( N \to \infty \)

\[
\text{Prob}[\ n \text{ events in } N \text{ throws}] = \left( \frac{N}{n} \right) p^n (1 - p)^{N-n} \to e^{-\lambda \frac{\lambda^n}{n!}} = \text{Prob}[n \text{ events in time } T]
\]

with \( pN \equiv \lambda = RT \). The waiting-time cumulative distribution function is

\[
S(\tau) \equiv \text{Prob}[\text{waiting time } \geq \tau] = \text{Prob}[0 \text{ events in time } \tau] = e^{-R\tau}
\]

\[
\Rightarrow D(\tau) = -\frac{dS(\tau)}{d\tau} = Re^{-R\tau}
\]
Clustering

- Fit of the scaling function: gamma distribution

\[ f(\theta) \propto \frac{1}{\theta^{0.3}} e^{-\theta/1.4} \quad \theta \equiv R \tau \]

Note that rescaling imposes \( \bar{\theta} = 1 \)
\[ \Rightarrow \] Only one parameter is independent

- The gamma distribution gives an increased probability for short waiting times
  (in comparison with a Poisson process, \( f(\theta) = e^{-\theta} \approx 1 \) for \( \theta < 1 \)) = clustering

\[ \Rightarrow \] Earthquakes tend to attract each other

\[ \Rightarrow \] Counterintuitive consequences:

\textit{The longer you have been expecting for an earthquake the longer you will still have to wait}
2. Waiting-Time Distributions & Scaling Laws: Clustering

Consequence of clustering: waiting-time paradox

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- **The longer you have been expecting for an earthquake the longer you will still have to wait**
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2. Waiting-Time Distributions & Scaling Laws: Clustering

Consequence of clustering: waiting-time paradox

- The longer you have been expecting for an earthquake, the longer you will still have to wait.
How is this paradoxical effect measured?

**Expected Residual Recurrence Time**

\[
\epsilon(\tau_0) \equiv \langle \tau - \tau_0 \mid \tau > \tau_0 \rangle = \frac{\int_{\tau_0}^{\infty} (\tau - \tau_0) D_w(\tau) d\tau}{\int_{\tau_0}^{\infty} D_w(\tau) d\tau} \approx \frac{\sum_{\forall i \text{ s.t. } \tau_i > \tau_0} (\tau_i - \tau_0)}{\text{num of equakes s.t. } \tau_i > \tau_0}
\]
• This result seems certainly **paradoxical**, as for example:
  - If you are **waiting for the metro**, you expect the next train is approaching
  - When you **celebrate your birthday**, you may feel you are consuming your life

• From a statistical point of view, this is **only counterintuitive**, as there are counterexamples
  - **Newborns** become “healthier” as time passes
  - **Companies** become more solid with time
    (it is not preferable to invest your money in a very new company!)

• For earthquakes, this seems even **more counterintuitive**:
  - The increase of time implies the **increase on stress** on the faults
  - The occurrence of earthquakes decreases the stress in some areas, but we have no occurrence since the last one
2. Universal scaling law for inter-event time distribution
2. Waiting-Time Distributions & Scaling Laws: Scaling

- **Scaling law**

\[ D(\tau, M_c) = R_c f(R_c \tau) \]

**Gutenberg-Richter law:** \( R_c \propto 10^{-b M_c} \)

In terms of energy: \( R_c \propto 1/E_c^\beta \), so:

\[ D(\tau, E_c) = E_c^{-\beta} \hat{f}(E_c^{-\beta} \tau) \]

This is the condition of scale invariance for 2d functions:

\[ F(x, y) = c F(x/a_1, y/a_2) \Rightarrow F(x, y) = x^\alpha f(y/x^\beta) \]

with \( f \) arbitrary, \( \alpha = \ln c / \ln a_1 \) and \( \beta = \ln a_2 / \ln a_1 \)

Why is this remarkable?
Relation with renormalization-group (RG) transformations
2. Waiting-Time Distributions & Scaling Laws: RG Transformations

Relation with renormalization-group transformations

Gala renormalizes into Lincoln!
• **RG transformation:**
The change of $M_c$ (decimation) plus the re-scaling with $R_c$ is analogous to a renormalization-group transformation

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**Christensen & Moloney**
*Complexity and Criticality*
2. Renormalization for Earthquakes: \( M \geq 5 \) for 1 year
2. Renormalization for Earthquakes: $M \geq 6$ for 1 year

![Graph showing earthquake magnitudes over time]
2. Renormalization for Earthquakes: $M \geq 6$ for 10 years
2. Renormalization for Earthquakes: $M \geq 5$ for 1 year

![Graph showing magnitude over time with a renormalization transformation applied]
2. Waiting-Time Distributions & Scaling Laws:

Mathematical results for renewal processes
(no correlations, magnitudes and times independent)

- Scaling $\Rightarrow$ Invariance of seismicity under RG transformations

- RG Transformation $=$ random thinning (decimation) + rescaling:

$$\top D(s) = \frac{pD(ps)}{1 - qD(ps)}$$

with $D(s)$ the Laplace transform of $D(\tau)$ and $p \equiv R(M'_c)/R(M_c)$, $q \equiv 1 - p$

- The only fixed point is Poisson $\top D(s) = D(s) \Rightarrow D(\tau) = Re^{-R\tau}$

- Moreover, the Poisson process is the attractor for all waiting time distributions
  with finite mean
2. Waiting-Time Distributions & Scaling Laws:

- Summary: for processes with no correlations and a finite mean
  \[ \Rightarrow \text{the Poisson process is a trivial fixed point} \]

- But the observed \( D(\tau) \) is not exponential
  \[ \Rightarrow \text{There must be correlations} \]

- Correlations are fundamental to determine the form of \( D(\tau) \)

- Short-range correlations do not seem enough to escape from Poisson

- Correlations should be long ranged (Lennartz et al., EPL 2008)

- A RG approach could provide scaling relations between the exponent of \( D(\tau) \), the exponent of correlations, and the Gutenberg-Richter exponent

- Connection with critical phenomena
Accumulated number of earthquakes (normalized) versus time with $N_{tot} = 84771$ for Southern California and $N_{tot} = 46054$ for worldwide seismicity.

⇒ worldwide seismicity is stationary, Southern California is not stationary
Changing the spatial region: WW stationary seismicity up to 2.8° (300 km)
Universal scaling law: also California, Japan, and Spain, for stationary periods.
ETAS model (epidemic-type aftershock sequence)

- Each earthquake ($i$) triggers other earthquakes with
  - Probability proportional to the Omori law, $1/(t - t_i)^p$
  - and proportional to the productivity law, $10^{aM_i}$ (power law)
  - Magnitude given by the Gutenberg-Richter law, $10^{-bM}$ (power law)

- Hard mathematics show that
  - The scaling function is different
  - Even more, a scaling law cannot hold exactly!

  ⇒ there must be (very) slow variations with magnitude
  ⇒ the ETAS distribution renormalizes to an exponential

Saichev & Sornette, PRL 2006
2. Waiting-Time Distributions & Scaling Laws: Criticism

- Saichev & Sornette’s fit: 4 parameters:

- But the ETAS model is not scale invariant (from its definition)!

- Try with a scale-invariant model?
  For instance: Vere-Jones model, AAP 2005;
  DS model, Lippiello et al. PRL 2007;
  BASS model, Turcotte et al. GRL 2007
2. Waiting-Time Distributions & Scaling Laws: Fractures

- Unexpected pulses detected in the CRESST project for dark matter search at the Gran Sasso Laboratory
  Cryogenic detector (at milliKelvin) made by a sapphire monocystal

> Åström et al. PLA 2006

- Radioactive contamination? No Poisson distribution!
- Origin: nanofractures in the crystal due to the tight clamping of the detector
2. Waiting-Time Distributions & Scaling Laws: Fractures

• Acoustic emission from laboratory rock fractures

![Graph showing probability density function](image)

☆ **Materials**: sandstones (wet conditions), granite (dry), Etna basalt (dry)
☆ **Loading conditions**: constant displacement rate, AE activity feedback control of loading, punch-through loading
☆ **Confined pressures**: from 5 to 100 MPa

Davidsen *et al.* PRL 2007
• Enormous range of validity of the scaling law:

★ From nanofractures involving the breaking of only several hundreds of covalent bonds (5 keV $\approx 8 \cdot 10^{-16}$ J)

to very large earthquakes ($M \geq 7$ or radiated energy $\geq 2 \cdot 10^{15}$ J)

⇒ More than 30 orders of magnitude of validity!

★ Profound differences between the homogeneity and regularity of a monocrystal at milli-Kelvin temperatures and the heterogeneity of fault gouge producing (and produced by) earthquakes

⇒ Universality
2. Waiting-Time Distributions & Scaling Laws: Correlations

**Conditional probability density** (for recurrence times)

\[ D_w(\tau|X) \equiv \frac{\text{Prob}[\tau \leq \text{recurrence time} < \tau + d\tau \text{ conditioned to } X]}{d\tau} \]

- For each \( \tau_i \), \( X \) will be different sets of values (large, small, etc.) of
  - \( M_i \) (current magnitude)
  - \( M_{i-1} \) (previous magnitude)
  - \( \tau_{i-1} \) (previous recurrence time)

- if \( D_w(\tau|X) = D_w(\tau) \Rightarrow \tau \text{ and } X \text{ independent} \)

- if \( D_w(\tau|X) \neq D_w(\tau) \Rightarrow \tau \text{ and } X \text{ correlated (linearly or nonlinearly)} \)
2. Waiting-Time Distributions & Scaling Laws: Correlations

![Diagram of a square grid with time on the horizontal axis and magnitude on the vertical axis. The grid shows the interdependence between different magnitudes and times.]

- \( M_{i-2} \)
- \( M_{i-1} \)
- \( M_i \)
- \( \tau_{i-1} \)
- \( \tau_i \)
Relation of $\tau_{i-1}$ with $\tau_i$ for Southern-California stationary seismicity

\[ D_w(\tau_i | \tau_a \leq \tau_{i-1} < \tau_b) \neq D_w(\tau_i) \implies \tau_i \text{ does depend on } \tau_{i-1} \] (positive correlation)
Relation of $M_{i-1}$ with $\tau_i$ for Southern-California stationary seismicity

$$D_w(\tau_i | M_{i-1} \geq M'_c) \neq D_w(\tau_i) \quad \Rightarrow \quad \tau_i \text{ does depend on } M_{i-1} \quad \text{(anticorrelation)}$$
Conclusion

- “The shorter the time between 2 earthquakes, the shorter the time to the next”
- “The larger the magnitude, the shorter the time to the next earthquake”

⇒ Recurrence times depend on history

- Possible existence of long-range correlations [Lennartz et al. 2007]
2. Waiting-Time Distributions & Scaling Laws: Correlations

\[ D_w(\tau_i | M_{i-1} \geq M'_c; M_c) \text{ only depends on } \tau \text{ and } M'_c - M_c \]

Moreover, a new scaling law holds, if \( R(M_c, M'_c) \equiv 1/\langle \tau(M_c, M'_c) \rangle \),

\[ \Rightarrow D_w(\tau_i | M_{i-1} \geq M'_c; M_c) = R_w f(R_w \tau_i, M'_c - M_c) \]
Relation of $M_i$ with $\tau_i$ for Southern-California stationary seismicity

\[ D_w(\tau_i | M_i \geq M'_c) \sim D_w(\tau_i) \implies M_i \text{ is independent on } \tau_i \]
Magnitude probability densities conditioned to the preceding magnitude

\[ D_w(M_i|M_{i-1} \geq M'_c) \approx D_w(M_i) \Rightarrow M_i \text{ is independent on } M_{i-1} \text{ (for } \tau_i > 30 \text{ min)} \]
Conclusion

- “The time you have been waiting for an earthquake does not influence its magnitude”

- “The magnitude of a given event does not influence the magnitude of the next one”

⇒ Magnitude seems to be independent on history

An earthquake does not know how big is going to be

But see Lippiello et al. PRL 2007
Verbs in novel *Clarissa*, by S. Richardson (year 1748, 1 million words)

⇒ Scaling and clustering (attraction)! Fit: gamma distribution with $\gamma = 0.6$
Comparing verbs in *Clarissa* with earthquakes in S. California, 1995-1998

\[
\theta = \ell / \ell_w \text{ for words, } \theta = R \tau = \tau / \bar{\tau} \text{ for earthquakes}
\]
Conclusions

• The dynamics of earthquake occurrence shows self-similar clustering, described by a universal scaling law.

• The same law holds for fractures up to very small scales (Davidsen et al., Åstrom et al.).

• The scaling law is equivalent to the invariance of the system under renormalization transformation.

• Correlations are essential to the existence of the scaling law.

• References at http://einstein.uab.es/acorralc.